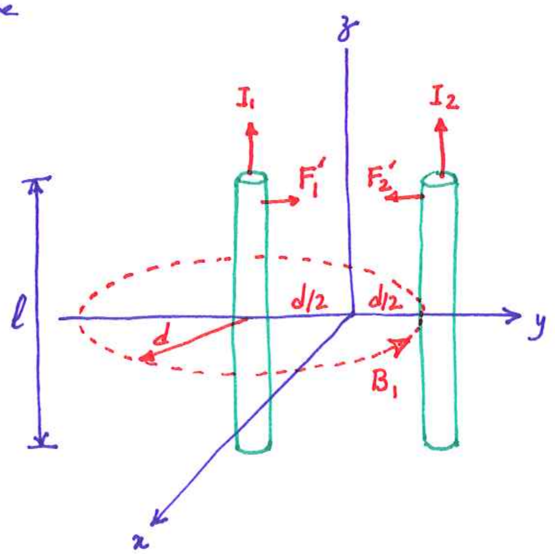
Magnetic Force between Two Parallel Conductors:

We now look into the magnetic force on a current carrying conductor when the conductor is in an external magnetic field. Let's consider two current-carrying conductors placed in each other's vicinity. Each generates a magnetic field that would apply force to the other conductor:

Using $\vec{B}$ derived for a wire that is long compared with the distance from the point at which B is measured:



$$\vec{B}_1 = -\hat{x}\frac{\mu_0 I_1}{2\pi d} \qquad \vec{F}_2 = I_2 \vec{\ell} \times \vec{B}_1$$

$$\vec{F}_2 = I_2 \ell\hat{z} \times \vec{B}_1 = I_2 \ell\, \hat{z} \times (-\hat{x}) \frac{\mu_0 I_1}{2\pi d}$$

$$= -\hat{y}\frac{\mu_0 I_1 I_2 \ell}{2\pi d}$$

The force per unit length is:

$$\vec{F}_2' = \frac{\vec{F}_2}{\ell} = -\hat{y}\frac{\mu_0 I_1 I_2}{2\pi d}$$ And similarly: $$\vec{F}_1' = \hat{y}\frac{\mu_0 I_1 I_2}{2\pi d}$$ → The two wires attract each other.

Maxwell's Magnetostatic Equations

Gauss's Law for Magnetism:

For electrostatics we had: $\vec{\nabla}\cdot\vec{D} = \rho_v \iff \oint_s \vec{D}\cdot\vec{ds} = Q$

For magnetostatics we have: $\boxed{\vec{\nabla}\cdot\vec{B} = 0 \iff \oint_s \vec{B}\cdot\vec{ds} = 0}$

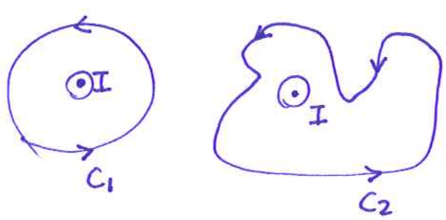
This relation is also called law of conservation of magnetic flux, or law of non-existance of isolated magnetic monopoles. In other word, magnetic field lines always form continuous closed loops.

Ampére's Law

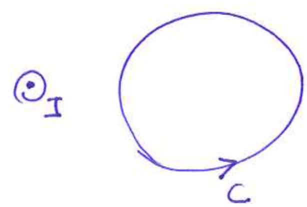
Amperes law comes from the Maxwell's equation of $\vec{\nabla}\times\vec{H}=\vec{J}$

$$\left.\begin{array}{l}\int_S(\vec{\nabla}\times\vec{H})\cdot ds = \overbrace{\int_S J\cdot ds}^{I} = I \\ \int_S(\vec{\nabla}\times\vec{H})\cdot ds = \oint_C \vec{H}\cdot\vec{dl}\end{array}\right\} \qquad \boxed{\oint_C H\cdot dl = I} \quad \text{(Ampére's law)}$$

So Ampere's law states that the line integral of current on a closed contour is equal to the current traversing the surface of the contour:



$$\oint_{C_1}\vec{H}\cdot\vec{dl} = \oint_{C_2}\vec{H}\cdot\vec{dl} = I$$

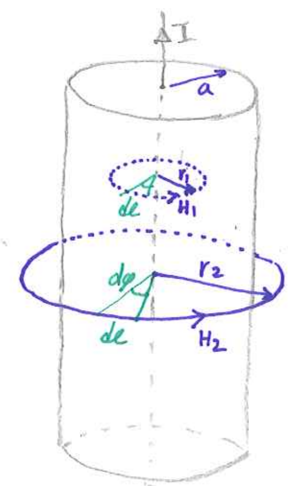


$\oint_C \vec{H}\cdot\vec{dl} = 0$ as there is no current crossing the surface.

Eample Magnetic flux of a long wire

Find the magnetic flux arround a wire with radius "a" and current I (a) inside the wire ($r\leq a$) (b) outside the wire ($r\geq a$).

$$\oint\vec{H}\cdot\vec{dl} = I \rightarrow \oint\vec{H_1}\cdot\vec{dl} = I_1$$



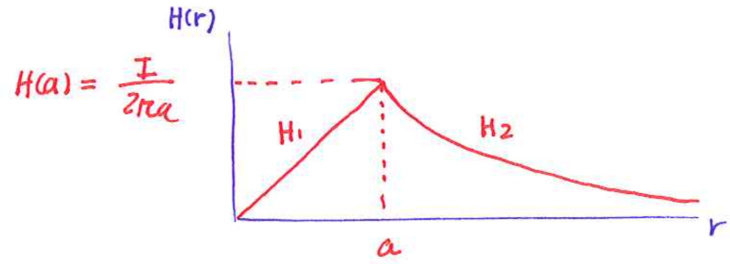
$$\left\{\begin{array}{l} I_1 = \int_{S_1}\vec{J}\cdot\vec{ds} = J\int_{S_1} ds = J\pi r_1^2 \quad J=\frac{I}{\pi a^2} \Rightarrow I_1 = I\frac{r_1^2}{a^2} \\ \oint\vec{H_1}\cdot\vec{dl} = \int_0^{2\pi}(H_1\hat{\varphi})\cdot(\hat{\varphi}\, r d\varphi) = 2\pi r_1 H_1 \end{array}\right.$$

$$\rightarrow \quad 2\pi r_1 H_1 = I\frac{r_1^2}{a^2} \rightarrow \vec{H_1} = \frac{r_1}{2\pi a^2} I\hat{\varphi} \qquad r_1\leq a$$

$$\oint \vec{H}_2 \cdot d\vec{l} = I_2 \qquad J_2 = I$$

$$\left.\oint \vec{H}_2 \cdot d\vec{l} = \int_0^{2\pi} (H_2\,\hat{\varphi})\cdot(\hat{\varphi}\, r_2\, d\varphi) = H_2\, 2\pi r_2 \right\} \quad H_2\, 2\pi r = I \rightarrow \vec{H}_2 = \frac{I}{2\pi r}\hat{\varphi} \qquad r_2 \geq a$$

This similar expression as we got using the Biot–Savart law.



Example Magnetic field inside a Toroidal Coil

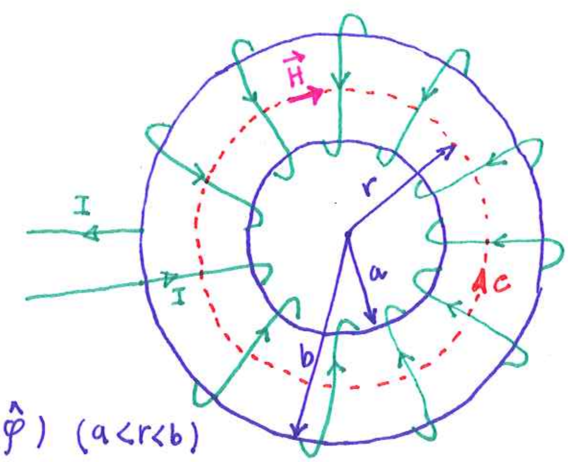
Calculate the magnetic field inside a toroidal coil (also called torus or toroid).

$$\vec{H} = -\hat{\varphi} H$$

$$\oint_C \vec{H}\cdot d\vec{l} = -NI$$ (we use ⊖ sign as the current is going into the surface)

$$\oint_C \vec{H}\cdot d\vec{l} = \int_0^{2\pi} (-H\hat{\varphi})\cdot(\hat{\varphi}\, r\, d\varphi)$$

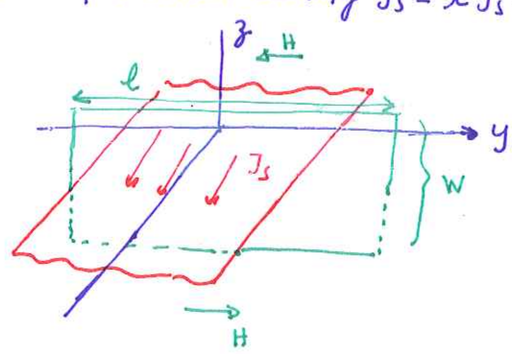
$$= -2\pi r H = -NI \rightarrow \vec{H} = \frac{NI}{2\pi r}(-\hat{\varphi}) \quad (a<r<b)$$



Example Magnetic Field of an infinite Current Sheet

Now consider a thin current sheet in the xy plane with surface current density $\vec{J}_s = \hat{x} J_s$.

$$H = \begin{cases} -\hat{y} H & \text{for } z>0 \\ y H & \text{for } z<0 \end{cases}$$



$$\oint_C \vec{H}\cdot d\vec{l} = \underbrace{\int_w \vec{H}\cdot d\vec{l}}_{=0} + \int_l \vec{H}\cdot d\vec{l} = Hl + Hl = 2Hl$$

$$I = J_s l$$

$$\Rightarrow \quad \vec{H} = \begin{cases} -\hat{y}\,\frac{J_s}{2} & \text{for } z>0 \\ \hat{y}\,\frac{J_s}{2} & \text{for } z<0 \end{cases}$$

Vector Magnetic Potential

In our discussion of electrostatic field we introduced the electrostatic potential V and defined $\vec{E} = -\vec{\nabla} V$. Here we wish to define a magnetic potential that results in magnetic flux B. Since $\vec{\nabla}\cdot\vec{B} = 0$, a good magnetic potential is such that $\vec{B} = \vec{\nabla}\times\vec{A}$ so that divergence of the curl is always zero. So we define the vector magnetic potential $\vec{A}$:

$$\boxed{\vec{B} = \vec{\nabla}\times\vec{A}} \quad (Wb/m^2)$$

Note that V was a scalar potential, but $\vec{A}$ is a vector.

we had $\vec{\nabla}\times\vec{H} = \vec{J} \rightarrow \vec{\nabla}\times\vec{B} = \mu\vec{J}$ (as $\vec{B} = \mu\vec{H}$).

If we replace $B = \nabla\times A \rightarrow \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \mu\vec{J}$

For any vector the laplacian is: $\nabla^2 A = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \vec{\nabla}\times(\vec{\nabla}\times\vec{A})$

$$= \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \mu\vec{J}$$

We chose $\vec{A}$ such that $\vec{B} = \vec{\nabla}\times\vec{A}$. However this has many solutions for A, unless we define another property for A. Here we choose $\vec{\nabla}\cdot\vec{A} = 0$ to simplify the above relation:

$$\text{Choose } \boxed{\vec{\nabla}\cdot\vec{A} = 0} \Rightarrow \boxed{\nabla^2 A = -\mu\vec{J}}$$

This is similar to the poisson's equ. $\nabla^2 V = -\frac{1}{\varepsilon}\rho$.

Note that in Cartesian coordinate: $\nabla^2 A = \frac{\partial^2 A}{\partial x^2}\hat{x} + \frac{\partial^2 A}{\partial y^2}\hat{y} + \frac{\partial^2 A}{\partial z^2}\hat{z} = \hat{x}\nabla^2 A_x + \hat{y}\nabla^2 A_y + \hat{z}\nabla^2 A_z$

$$\Rightarrow \quad \nabla^2 A_x = -\mu J_x$$

$$\nabla^2 A_y = -\mu J_y$$

$$\nabla^2 A_z = -\mu J_z$$

This is very similar to the poisson's equation:

$$\nabla^2 V = -\frac{\rho}{\varepsilon} \longrightarrow V = \frac{1}{4\pi\varepsilon}\int_V \frac{\rho}{R}\, dv$$

$$\nabla^2 A_x = -\mu J_x \longrightarrow A_x = \frac{\mu}{4\pi}\int_V \frac{J_x}{R}\, dv \qquad (wb/m)$$

And similarly for A_y and A_z. So we can write:

$$\boxed{\vec{A} = \frac{\mu}{4\pi}\int_V \frac{\vec{J}}{R}\, dv} \qquad (wb/m)$$

If the current is on a surface with surface current density, $\vec{J}dv$ is replaced with $\vec{J}_s ds$ and V by S. Same if it's a line current density, $\vec{J}dv$ is replaced with $\vec{J}_l dl$ and V by l. The above relation, is another approach to calculat the magnetic field resulted from a current (besides the Amper's law and Biot-Savart).

Magnetic flux, Φ

Magnetic flux Φ is defined as: $\boxed{\Phi = \int_S \vec{B}\cdot\vec{ds}}$ (wb) $\vec{B}$ is mag. flux density.

$$\rightarrow \Phi = \int_S (\vec{\nabla}\times\vec{A})\cdot ds = \oint_C \vec{A}\cdot\vec{dl} \ (wb) \rightarrow$$

$$\boxed{\Phi = \oint_C \vec{A}\cdot\vec{dl}} \qquad (wb)$$

Magnetic Properties of Materials

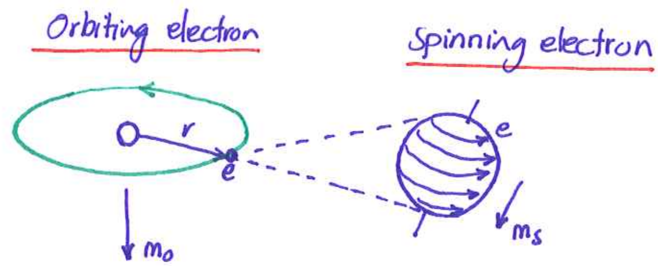
Materials are classified in terms of their magnetic properties as:

Diamagnetic: Atoms have no permanent magnetic moments. (Bi, Cu, Au, Pb, Ag, Si, Hg, diamond)

Paramagnetic: Atoms have permanent magnetic moments, with no hystersis (Al, Ca, Cr, Mg, Pt, W, Nb)

Ferromagnetic: Atoms have permanent magnetic moments and have hystersis (Fe, Ni, Co)

Orbital and Spin Magnetic Moments



In a classical picture, electron rotates in a circular orbit of radius r with a constant velocity u.

one complete rotation takes $T = \frac{2\pi r}{u}$ time.

This constitutes a tiny current loop with I as:

$$I = -\frac{e}{T} = -\frac{eu}{2\pi r}$$

The magnitude of the orbital magnetic moment m_o is:

$$m_o = IA = \left(-\frac{eu}{2\pi r}\right)(\pi r^2) = -\frac{eur}{2} = -\left(\frac{e}{2m_e}\right)\overbrace{m_e u r}^{L_e\text{ : angular momentum of electron with mass } m_e}$$

$$m_o = -\left(\frac{e}{2m_e}\right)L_e$$

According to quantum physics, the angular momentum is quantized: L_e is always an integer multiple of $\hbar = \frac{h}{2\pi}$: $L_e = 0, \hbar, 2\hbar, 3\hbar, \ldots$ So the smallest non-zero orbital magnetic moment is:

$$\boxed{m_o = -\frac{e\hbar}{2m_e}}$$

Although all materials have electrons and electrons have magnetic moment dipole moments, most materials are effectively nonmagnetic. This is because the electron magnetic moments in absence of external $\vec{B}$ are randomly oriented and cancel each other's effect.

In addition to angular magnetic moments electrons generate a spin magnetic moment, m_s.

classically, this is due to its spinning motion about its own axis:

The magnitude of m_s predicted from quantum physics is:

$$\boxed{m_s = -\frac{e\hbar}{2m_e}}$$

which equal to the minimum orbital magnetic moment m_0. Atoms with even number of electrons usually have electrons with pairs of having opposite spin directions, therefore cancel each other's spin magnetic moments. If the number of electrons is odd, the atom will have a nonzero spin magnetic moment.

The nucleus of an atom also has a spinning motion (classically) and a spin magnetic moment. But as the mass of nucleus is much larger than that of electron, its spin magnetic moment is very smaller (on the order of 10^{-3} of that of electron)

Magnetic Permeability

Magnetization vector M of a material is the sum of all magnetic dipole moments of the atoms contained in a unit volume of the material. The corresponding magnetic flux density is: $\vec{B}_m = \mu_0 \vec{M}$. Hence in presence of an external $\vec{H}$, the total magnetic flux density is:

$$\vec{B} = \mu_0 \vec{H} + \mu_0 \vec{M} = \mu_0 (\vec{H} + \vec{M})$$

In general, the material is magnetized in response to the external $\vec{H}$. So $\vec{M} = \chi_m \vec{H}$

where χ_m is dimensionless and called magnetic susceptibility. In diamagnetic and paramagnetic materials χ_m is constant so M is a linear function of H. But in ferroelectric materials χ_m also depends on the history of the material. Also χ_m depends on the magnitude of H; hence, M is a non-linear function of H.

We can now write for B:

$$\vec{B} = \mu_0 \vec{H} + \mu_0 \vec{M} = \mu_0 \vec{H} + \mu_0 \chi_m \vec{H} = \overbrace{\mu_0 (1+\chi_m)}^{\mu} \vec{H}$$

or: $\vec{B} = \mu \vec{H}$ where μ is the magnetic permeability give by:

$$\boxed{\mu = \mu_0 (1+\chi_m)} \quad (H/m)$$

We can also define the relative permeability μ_r:

$$\boxed{\mu_r = \frac{\mu}{\mu_0} = 1 + \chi_m}$$

Diamagnetic materials have negative susceptibilities and paramagnetic materials have positive χ_m. However magnitude of χ_m is very small in both materials in the order of 10^{-5}. So $\chi_m \ll 1 \Rightarrow \mu_r \simeq 1$ or $\mu \simeq \mu_0$ for dia- and paramagnetics, which includes dielectrics and most metals.

For ferroelectrics $|\mu_r| \gg 1$. For example for pure Iron $\mu_r \simeq 2 \times 10^5$.